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# On the complexity of determining tolerances for $\epsilon$ -optimal solutions to min-max combinatorial optimization problems

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## Abstract

Sensitivity analysis of  $\epsilon$ -optimal solutions is the problem of calculating the range within which a problem parameter may lie so that the given solution remains  $\epsilon$ -optimal. In this paper we study the sensitivity analysis problem for  $\epsilon$ -optimal solutions to combinatorial optimization problems with min-max objectives where  $\epsilon > 0$ . We show that the problem is easy if the original problem is easy. We also show that the converse is true under the assumption that it is possible to calculate an  $\epsilon$ -optimal solution to the problem in polynomial time.

*Keywords: tolerances, complexity, combinatorial optimization, min-max problems,  $\epsilon$ -optimality*

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A study of the effect of changes in problem data on optimal solutions to optimization problems is an important step in gaining insight into the problem itself, and can be carried out in a variety of ways. One can study the effect of changing a single parameter (sensitivity analysis), or of changing multiple parameters simultaneously (stability analysis). One may observe the changes in an optimal solution when a problem parameter changes from its lowest possible value to its highest possible value (parametric analysis), or find the range within which a parameter must lie for the optimal solution to remain optimal (tolerance analysis). Another approach, although less popular, is to compute the set of  $k$  best solutions for the problem ( $k$ -best approach).

Sensitivity analysis of combinatorial optimization problems (COPs), although not as popular as that of linear optimization, has been widely studied in the last thirty years. Initial studies considered them as mere special cases of general integer programming problems and used the parametric analysis approach (refer, for e.g. Nauss [3]). In the 1980's sensitivity analysis of individual COPs received attention, although the approach was still primarily parametric. In the 1990's, the tolerance approach gained favour and the sensitivity analysis of general COPs were studied. Stability analysis of individual COP's was also studied extensively. For a survey on results in this field prior to 1977, we refer the reader to Nauss [3]. Greenberg [2] provides an extensive and annotated bibliography of 136 publications in this field published after 1977. An updated version of this reference is available at [www.cudenver.edu/~hgreenbe/aboutme/pubrec.html](http://www.cudenver.edu/~hgreenbe/aboutme/pubrec.html).

A general COP can be defined as follows. It is a collection of problem instances  $\Pi = (G, \mathcal{S}, z)$ , where  $\mathcal{S} \subseteq 2^G$ , and  $z : \mathcal{S} \rightarrow \mathbb{R}$ . The finite set  $G$  is called the ground set. Each element  $e \in G$  has a cost  $c_e$ . The members of  $\mathcal{S}$  are called feasible solutions, and  $z$  is referred to as the objective function to be minimized. Without loss of generality we will assume that  $z(\emptyset) = \infty$ .

We denote the problem of finding an optimal solution, i.e. a member of  $\arg \min\{z(S) : S \in \mathcal{S}\}$  for a given instance  $\pi \in \Pi$  by  $\text{OPT}(\Pi)$ . For any  $\epsilon > 0$  we denote by  $\text{OPT}_\epsilon(\Pi)$  the problem of finding an  $\epsilon$ -optimal solution, i.e. a member of  $\{S : S \in \mathcal{S}, |z(S) - z(S^*)| \leq \epsilon z(S^*)\}$  for a given instance  $\pi \in \Pi$ , where  $S^*$  is an optimal solution.

A COP  $\Pi$  is said to have a min-max objective if  $z(S) = \max\{c_e : e \in S\}$  for every  $S \in \mathcal{S}$  and each  $\pi \in \Pi$ . We refer to such COPs as min-max COPs.

The sensitivity analysis problem of a COP using the tolerance approach involves finding for each parameter, upper and lower bounds within which the value of the parameter can vary for the optimal solution to remain optimal. The complexity of this problem was studied independently by Van Hoesel & Wagelmans [5] and Ramaswamy

& Chakravarti [4]. Van Hoesel & Wagelmans dealt with COPs with min-sum objectives in which the parameters could only have non-negative values. Ramaswamy & Chakravarti, on the other hand, considered both the min-sum and the min-max objectives, and their results are valid even when the parameters assume negative values. The results obtained by both were similar, viz. the sensitivity analysis problem is polynomially solvable if and only if the original COP is. Chakravarti & Wagelmans [1] proved that the stability analysis of both optimal and  $\epsilon$ -approximate solutions to COPs with min-sum objectives and of optimal solutions to COPs with min-max objectives is polynomial if the original COP is polynomial.

In [5] Van Hoesel & Wagelmans defined the sensitivity analysis of  $\epsilon$ -optimal solutions to general COPs as follows.

Problem  $\text{SA}\epsilon(\Pi)$ : Sensitivity analysis of  $\epsilon$ -optimal solutions of a given COP  $\Pi$

Input Instance  $\pi = (G, S)$  of  $\Pi$ ,  $\epsilon > 0$ ,  $\epsilon$ -optimal solution  $S^\epsilon$  to  $\pi$ .

Output For each  $e \in G$

Upper tolerance limit  $\beta_e = \sup\{\delta \in \mathbb{R}: S^\epsilon \text{ remains } \epsilon\text{-optimal when } c_e \rightarrow c_e + \delta\}$ ,

Lower tolerance limit  $\alpha_e = \sup\{\delta \in \mathbb{R}: S^\epsilon \text{ remains } \epsilon\text{-optimal when } c_e \rightarrow c_e - \delta\}$ .

In [5] the complexity of this problem for COPs with min-sum objectives is analysed and it is shown that  $\text{SA}\epsilon(\Pi)$  is easy if and only if  $\text{OPT}(\Pi)$  is easy.

In this paper we determine the complexity of sensitivity analysis for  $\epsilon$ -optimal solutions to min-max COPs. It is clear from the discussions in Ramaswamy & Chakravarti [4] that arguments used to study complexity of sensitivity analysis for min-sum problems do not translate automatically to arguments for min-max problems, due to the difference in the nature of the objective functions in the two cases.

We will use the following notation in the remainder of the paper.  $\mathcal{P}$  will denote the set of polynomially solvable optimization problems. The COP at hand will be denoted by  $\Pi$ .  $\pi$  will denote an instance of  $\Pi$ . If the value of any problem element of  $\pi$  is changed by a cost transformation  $T$ , the new instance will be denoted by  $\pi_T$ .  $S^*$  will denote an optimal solution to  $\pi$  and  $S^\epsilon$  an  $\epsilon$ -optimal solution. Given an element  $e \in G$ ,  $S^\epsilon$  will denote an element of the set  $\{S : S \in S, e \notin S\}$ . Given  $e \in S \in S$ , we call  $S$   $e$ -critical with respect to  $z$  if  $z(S) = c_e$ . Given a solution  $S = \{e_{[1]}, e_{[2]}, \dots\}$  with  $c_{e_{[1]}} \geq c_{e_{[2]}} \geq \dots$ , the cost of the second largest element of  $S$ , i.e.  $c_{e_{[2]}}$  will be denoted by  $c_{[2]}(S)$ . If  $S$  is a singleton, then  $c_{[2]}(S)$  is assumed to be  $\infty$ . Note that  $c_{[2]}(S)$  may equal  $z(S)$ .

Let us first assume that  $\text{OPT}(\Pi) \in \mathcal{P}$  and that the polynomial algorithm  $\mathcal{A}$  solves any instance of  $\Pi$ . Note that for any  $e \in G$  for any instance  $\pi$  of  $\Pi$ ,  $S^\epsilon$  can be found in polynomial time by setting  $c_e \rightarrow \infty$ . In the following lemma, we are concerned with

the complexity of finding the cost of a smallest second largest element in any  $e$ -critical solution, when  $c_e$  is at least as large as  $z(S^*)$ .

**Lemma 1** *If  $\text{OPT}(\Pi) \in \mathcal{P}$ , then given an instance  $\pi$  of  $\Pi$  and  $e \in G$  with  $c_e \geq z(S^*)$ , we can determine in polynomial time, the value of  $c_{[2]}(S)$  for any solution  $S$  satisfying the following criteria, or deduce that no such solution exists.*

1.  $S$  is  $e$ -critical.
2.  $c_{[2]}(S) < z(S^*)$ .
3.  $\nexists S' \in \mathcal{S}$  with  $c_{[2]}(S') < c_{[2]}(S)$  satisfying 1 and 2.

PROOF. Since  $\text{OPT}(\Pi) \in \mathcal{P}$ , we can calculate  $z(S^*)$  in polynomial time. Let us transform  $\pi$  to the instance  $\pi_T$  as follows:  $G \rightarrow G \setminus \{e\}$  and  $S \rightarrow S \setminus \{e\}$  for every  $S \in \mathcal{S}$ . This transformation only affects the cost of  $e$ -critical solutions satisfying  $S \setminus \{e\} \neq \emptyset$ , each of which now has a cost equal to the cost of its second largest element. If any solution satisfies  $S \setminus \{e\} = \emptyset$ , then its cost becomes infinite after the transformation.

Let us now apply Algorithm  $\mathcal{A}$  on  $\pi_T$ . If the objective value of the output is equal to that of the original problem, then it is obvious that no solution satisfying the three criteria exists. On the other hand, if the objective value of the optimal solution to  $\pi_T$  is lower, then the new objective value is the required output, since it is the lowest  $c_{[2]}(\cdot)$  value among all solutions satisfying conditions 1 and 2 in  $\pi$ .

The method mentioned above involves invoking Algorithm  $\mathcal{A}$  twice, and transforming a problem instance. Since all the operations are polynomial (we do not explicitly transform individual solutions) the method itself is polynomial time.  $\square$

**Theorem 1** *Let  $\epsilon > 0$ ,  $\Pi = (G, \mathcal{S}, z)$  be a min-max COP. Then  $\text{OPT}(\Pi) \in \mathcal{P} \Rightarrow \text{SA}\epsilon(\Pi) \in \mathcal{P}$ .*

PROOF. Since  $S^\epsilon$  is known, we can predict its behavior when  $c_e$  changes. Also since  $\text{OPT}(\Pi) \in \mathcal{P}$ , an optimal solution  $S^*$  to an instance  $\pi$  of  $\Pi$  can be calculated in polynomial time.

We will first show that if  $\text{OPT}(\Pi) \in \mathcal{P}$ , then given an  $\epsilon$ -optimal solution  $S^\epsilon$  and  $e \in G$ ,  $\beta_e$  can be calculated in polynomial time. If  $e \notin S^*$ , then the optimal objective value is not affected by an increase in  $c_e$ . If  $e \in S^*$ , the optimal objective value remains  $z(S^*)$  until  $c_e$  exceeds  $z(S^\epsilon)$ . After that, the new optimal objective value remains constant at  $z(S^\epsilon)$ . So we see that we can deduce the response of the optimal objective value

to changes in  $c_e$ . We know that  $z(S^\epsilon)$  can be calculated in polynomial time. Since we know the responses of both  $S^\epsilon$  and the optimal objective value to changes in  $c_e$  in polynomial time,  $\beta_e$  can be calculated in polynomial time.

Finally we show that if  $\text{OPT}(\Pi) \in \mathcal{P}$ , then given an  $\epsilon$ -optimal solution  $S^\epsilon$  and  $e \in G$ ,  $\alpha_e$  can be calculated in polynomial time. If  $c_e$  decreases, then  $S^*$  can become suboptimal only if  $c_e > z(S^*)$ , and there exists an  $e$ -critical solution satisfying the three conditions in Lemma 1. In that case, the optimal objective value remains  $z(S^*)$  until  $c_e$  reduces to  $z(S^*)$ , then becomes  $c_e$  until  $c_e$  reduces to  $c_{[2]}(S)$  for a solution  $S$  satisfying the three conditions in Lemma 1, and then remains constant at  $c_{[2]}(S)$ . According to Lemma 1, we can check for the existence of such a solution  $S$  and find  $c_{[2]}(S)$  if it exists, in polynomial time. Therefore we can predict the response of the optimal objective value to changes in  $c_e$ . Since we know the responses of both  $S^\epsilon$  and the optimal objective value to changes in  $c_e$  in polynomial time,  $\alpha_e$  can be calculated in polynomial time.  $\square$

In the remainder of the paper, we assume that  $\text{SA}\epsilon(\Pi) \in \mathcal{P}$  and make the further assumption that  $\text{OPT}\epsilon(\Pi) \in \mathcal{P}$ . Under these assumptions we show that an optimal solution to any instance  $\pi$  of  $\Pi$  can be calculated in polynomial time.

The following two lemmas provide a polynomial time characterization of the objective value of an optimal solution.

**Lemma 2** *Given  $\epsilon > 0$ , an  $\epsilon$ -optimal solution  $S^\epsilon$  to an instance  $\pi \in \Pi$ , and an element  $e \in S^\epsilon$  such that  $Z(S^\epsilon) = c_e$ ,  $\beta_e \geq \epsilon z(S^\epsilon) \iff z(S^*) = z(S^\epsilon)$  for any optimal solution  $S^*$  to  $\pi$ .*

**PROOF.**  $(\Rightarrow)$  Assume to the contrary, that there exists an optimal solution  $S_1^*$  with  $z(S_1^*) < z(S^\epsilon)$ . Then  $e \notin S_1^*$ , and the value of  $z(S_1^*)$  is not affected by an increase in  $c_e$  while that of  $S^\epsilon$  increases with a slope of 1.

Hence  $\beta_e = (1 + \epsilon)z(S^*) - z(S^\epsilon) < (1 + \epsilon)z(S^\epsilon) - z(S^\epsilon) = \epsilon z(S^\epsilon)$  which is a contradiction.

$(\Leftarrow)$  We distinguish between the following two cases:

1.  $e \in S^* \forall S^*$
2.  $\exists S^*$  such that  $e \notin S^*$

In case 1, if the value of  $c_e$  increases, both  $z(S^\epsilon)$  and  $z(S^*)$  increase with a slope of 1 until they reach the value  $z(S^\epsilon)$ . After that,  $z(S^\epsilon)$  keeps increasing with slope 1, but  $z(S^*)$  remains constant at  $z(S^\epsilon)$ . Hence  $\beta_e = (z(S^\epsilon) - z(S^*)) + \epsilon z(S^\epsilon) \geq \epsilon z(S^\epsilon)$ .

In case 2, The objective value of the optimal solution is not affected by an increase in the value of  $c_e$  but  $z(S^\epsilon)$  increases with a slope of 1. So clearly  $\beta_e = \epsilon z(S^\epsilon)$ .  $\square$

**Lemma 3** *Given  $\epsilon > 0$ , an  $\epsilon$ -optimal solution  $S^\epsilon$  to an instance  $\pi \in \Pi$ , and an element  $e \in S^\epsilon$  such that  $Z(S^\epsilon) = c_e$ ,  $\beta_e < \epsilon z(S^\epsilon) \Rightarrow z(S^*) = \frac{c_e + \beta_e}{1 + \epsilon}$  for any optimal solution  $S^*$  to  $\pi$ .*

**PROOF.** It follows from Lemma 2 that  $\beta_e < \epsilon z(S^\epsilon) \Rightarrow z(S^*) < z(S^\epsilon)$ . Therefore  $e \notin S^*$  which implies that the objective value of the optimal solution is not affected by an increase in the value of  $c_e$  but  $z(S^\epsilon)$  increases with a slope of 1. So  $z(S^\epsilon) + \beta_e = c_e + \beta_e = (1 + \epsilon)z(S^*)$ . The result follows.  $\square$

**Theorem 2** *Let  $\epsilon > 0$  and  $\Pi = (G, \mathcal{S}, z)$  be a min-max COP. Then  $SA\epsilon(\Pi), OPT\epsilon(\Pi) \in \mathcal{P} \Rightarrow OPT(\Pi) \in \mathcal{P}$ .*

**PROOF.** From Lemmas 2 and 3 we know that if  $SA\epsilon(\Pi), OPT\epsilon(\Pi) \in \mathcal{P}$ , then the objective value of an optimal solution can be calculated in polynomial time. This is equivalent to saying that the evaluation version of  $\Pi$  is polynomially solvable.

It is common knowledge that if the evaluation version of a COP is polynomially solvable, so is the optimization version. For the sake of completeness, we present the following Algorithm  $\mathcal{B}$  that generates an optimal solution to an instance  $\pi$  of  $\Pi$ , given a polynomial time algorithm to calculate an  $\epsilon$ -optimal solution to  $\pi$ .

Algorithm  $\mathcal{B}$

Input: An instance  $\pi$  of  $\Pi$ , an algorithm to calculate an  $\epsilon$ -optimal solution to  $\pi$ .

Output: An optimal solution to  $\pi$ .

begin

$s \leftarrow \emptyset$ ;

Obtain an  $\epsilon$ -optimal solution  $S^\epsilon$  to  $\pi$ ;

Obtain  $z^*$ , the optimal objective value of  $\pi$ ;

if  $z^* = z(S^\epsilon)$  then

Return  $S^\epsilon$  and stop;

Arrange elements  $e \in G$  in non-increasing order of  $c_e$  values;

/\* the elements will be chosen in this order in the following for loop. \*/

```

for each  $e \in G$  do
begin
    Apply  $T : c_e \leftarrow \infty$ ;
    Obtain an  $\epsilon$ -optimal solution  $S^\epsilon$  to  $\pi_T$ ;
    Obtain  $z^T$ , the optimal objective value of  $\pi_T$ ;
    if  $z^T < z^*$  then
    begin
         $S \leftarrow S \cup \{e\}$ ;
        Restore the value of  $c_e$ ;
    end
end
Return  $S$ ;
end.

```

It is trivial to see that Algorithm  $\mathcal{B}$  is correct. Apart from a (polynomial) sorting operation, the algorithm calculates  $\epsilon$ -optimal solutions and uses them to calculate the optimal objective value  $\mathcal{O}(|G|)$  times. According to Lemmas 2 and 3, the latter operation is polynomial time. Hence Algorithm  $\mathcal{B}$  is also polynomial algorithm.  $\square$

In Theorem 1 we proved that the sensitivity analysis problem for COPs with min-max objective functions can be done in polynomial time if the original COP can be solved in polynomial time. In Theorem 2 we showed that under the additional weak assumption that an  $\epsilon$ -optimal solution can be calculated in polynomial time, then if the sensitivity analysis problem for a COP is polynomially solvable, so is the original COP. Hence we deduce that the sensitivity analysis problem for  $\epsilon$ -optimal solutions to combinatorial problems with min-max objectives is as difficult as the original problems themselves. In this respect, our results complement similar research on min-sum problems and also research on optimal solutions to both min-sum and min-max problems.

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